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NOTE ON MULTIPLE q-ZETA FUNCTIONS

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Abstract. In this paper we consider the analytic continuation of the multiple Euler q-zeta function in the complex number field as follows:

$$\zeta_{r,q}^{E}(s,x)=[2]_q^r\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(-1)^{m_1+\cdots+m_r}}{[x+m_1+\cdots+m_r]_q^s},$$

where $q\in\mathbb{C}$ with $|q|<1$, $\Re(x)>0$, and $r\in\mathbb{N}$. Thus, we investigate their behavior near the poles and give the corresponding functional equations.

1. Introduction/ Preliminaries

Let $\mathbb{C}$ be the complex number field. For $s\in\mathbb{C}$, the Hurwitz's type Euler zeta function is defined by

$$(1)\qquad \zeta^{E}(s,x)=2\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+z)^s},\ \text{where } s\in\mathbb{C},\ z\neq 0,-1,-2,\cdots,\ (\text{ see [11] }).$$

Thus, we note that $\zeta^E(s,x)$ is a meromorphic function in whole complex s-plane. It is well known that the Euler polynomials are defined as

$$(2)\qquad \frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\ \text{for } |t|<\pi,$$

and $E_n=E_n(0)$ are called the n-th Euler numbers (see [7, 8, 9, 11]). By (1) and (2), we note that $\zeta^E(-n,x)=E_n(x)$, for $n\in\mathbb{Z}_+$. Throughout this paper we assume

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that $q \in \mathbb{C}$ with $|q| < 1$ and we use the notation of q-numbers as $[x]_q = \frac{1-q^x}{1-q}$. The q-Euler numbers are defined as

$$(3) \qquad E_{0,q} = \frac{2}{[2]_q}, \text{ and } (qE+1)^n + E_{n,q} = 0 \text{ if } n \geq 1 \ ,$$

where we use the standard convention about replacing E^k by $E_{k,q}$(see [7]). Thus, we define the q-Euler polynomials as follows:

$$(4) \qquad E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} q^{lx} E_{l,q}, \text{ (see [7, 8, 15]).}$$

For $s \in \mathbb{C}$, the q-extension of Hurwitz's type q-Euler zeta function is defined by

$$(5) \qquad \zeta_q^E(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n}{[n+x]_q^s}, \text{ where } x \neq 0, -1, -2, \cdots.$$

For $n \in \mathbb{Z}_+$, we have $\zeta_q^E(-n,x) = E_{n,q}(x)$(see [6, 7, 15]). Let χ be a Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1(mod\ 2)$. It is known that the generalized q-Euler polynomials attached to χ are defined by

$$(6) \qquad F_{q,\chi}(t,x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) e^{[m+x]_q t} = \sum_{m=0}^{\infty} E_{m,\chi,q}(x) \frac{t^m}{m!}, \text{ see [7] .}$$

Note that

$$\lim_{q \to 1} F_{q,\chi}(t,x) = \frac{2\sum_{a=1}^{f-1}(-1)^a \chi(a) e^{at}}{e^{ft}+1} e^{xt} = \sum_{m=0}^{\infty} E_{m,\chi}(x) \frac{t^m}{m!},$$

where $E_{m,\chi}(x)$ are called the m-th generalized Euler polynomials attached to χ. From (6), we can derive the following equation.

(7)

$$E_{n,\chi,q}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) [m+x]_q^n = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-q^x)^l \sum_{a=0}^{f-1} \frac{(-1)^a \chi(a) q^{la}}{1+q^{lf}}.$$

Now, we consider the Dirichlet's type Euler q-l-function which interpolate $E_{n,\chi,q}(x)$ at negative integer. For $s \in \mathbb{C}$, define

$$l_q(s,x|\chi) = \sum_{n=0}^{\infty} \frac{\chi(n)(-1)^n}{[n+x]_q^s}, \ x \neq 0, -1, -2, \cdots, \text{ (see [6, 7, 8, 15]).}$$

Note that $l_q(-n,x|\chi) = E_{n,\chi,q}(x)$ for $n \in \mathbb{Z}_+$. In the special case $x = 0$, $E_{n,\chi,q}(= E_{n,\chi,q}(0))$ are called the n-th generalized Euler numbers attached to χ. The theory of quantum groups has been quite successful in producing identities for q-special function. Recently, several mathematicians have studied q-theory in the several areas(see [1-23]). In this paper we approach the q-theory in the area of special function. That is, we first consider the analytic continuation of multiple q-Euler zeta function in the complex plane as follows:

$$\zeta^E_{r,q}(s,x) = [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}}{[x+m_1+\cdots+m_r]^s_q}, \ s\in\mathbb{C},\ x\neq 0,-1,\cdots. \tag{8}$$

From (8), we investigate some identities for the multiple q-Euler numbers and polynomials. Finally, we give interesting functional equation related to the multiple q-Euler polynomials, gamma functions and multiple q-Euler zeta function.

2. Multiple q-Euler polynomials and multiple q-Euler zeta functions

From (3), we note that

$$E_{n,q} = \frac{[2]_q}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-q^x)^l}{(1+q^l} = [2]_q\sum_{m=0}^{\infty}(-1)^m[m+x]^n_q. \tag{9}$$

Let $F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}$. Then we see that

$$F_q(t,x) = [2]_q\sum_{m=0}^{\infty}(-1)^m e^{[m+x]_q t}. \tag{10}$$

From (10), we note that $\lim_{q\to 1} F_q(t,x) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$, where $E_n(x)$ are called the n-th Euler polynomials. For $s\in\mathbb{C}$, we have

$$\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}F_q(-t,x)dt = [2]_q\sum_{n=0}^{\infty}\frac{(-1)^n}{[n+x]^s_q}, \ \text{where } x\neq 0,-1,-2,\cdots. \tag{11}$$

By Cauchy residue theorem and Laurent series, we see that $\zeta^E_q(-n,x) = E_{n,q}(x)$ for $n\in\mathbb{Z}_+$. Let χ be the Dirichlet's character with conductor $f(=odd)\in\mathbb{N}$. From (6), we can derive

$$\begin{aligned} F_{q,\chi}(t,x) &= [2]_q\sum_{a=0}^{f-1}(-1)^a\chi(a)\sum_{n=0}^{\infty}(-1)^n e^{[a+x+nf]_q t} \\ &= [2]_q\sum_{a=0}^{f-1}(-1)^a\chi(a)\sum_{n=0}^{\infty}(-1)^n e^{[f]_q[\frac{x+a}{f}+n]_{q^f} t}. \end{aligned} \tag{12}$$

Let us define the Dirichlet's type q-Euler l-function as follows:

$$l_q(s,x|\chi) = [2]_q\sum_{m=0}^{\infty}\frac{\chi(m)(-1)^m}{[m+x]^s_q}, \ \text{where } s\in\mathbb{C},\ x\neq 0,-1,-2,\cdots. \tag{13}$$

From the Mellin transformation of $F_{q,\chi}(t,x)$, we note that

(14)
$$\frac{1}{\Gamma(s)}\int_0^\infty F_{q,\chi}(-t,x)t^{s-1}dt=[2]_q\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)}{[n+x]_q^s},\ \text{where } s\in\mathbb{C},\ x\neq 0,-1,-2,\cdots.$$

By Laurent series and Cauchy residue theorem, we see that $l_q(-n,x|\chi)=E_{n,\chi,q}(x)$ for $n\in\mathbb{Z}_+$. Let us consider the following q-Euler polynomials of order $r(\in\mathbb{N})$.

$$F_q^{(r)}(t,x)=[2]_q^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}e^{[m_1+\cdots+m_r+x]_qt}=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{15}$$

In the special case $x=0$, $E_{n,q}^{(r)}(=E_{n,q}^{(r)}(0))$ are called the n-th q-Euler numbers of order r. It is easy to show that $\lim_{q\to 1}F_q^{(r)}(t,x)=\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}$, where $E_n^{(r)}(x)$ are called the n-th Euler polynomials of order r. From (15), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!}&=[2]_q^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}e^{[m_1+\cdots+m_r+x]_qt}\\&=[2]_q^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m e^{[m+x]_qt}.\end{aligned} \tag{16}$$

Thus, we have

$$E_{n,q}^{(r)}(x)=\frac{[2]_q^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(\frac{1}{1+q^l}\right)^r.$$

Therefore, we obtain the following proposition.

Proposition 1. *For $r\in\mathbb{N}$, $n\in\mathbb{Z}_+$, we have*

$$\begin{aligned}E_{n,q}^{(r)}(x)&=[2]_q^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n\\&=[2]_q^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m[m+x]_q^n\\&=\frac{[2]_q^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(\frac{1}{1+q^l}\right)^r.\end{aligned}$$

By Mellin transformation of $F_q^{(r)}(t,x)$, we see that

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty F_q^{(r)}(-t,x)t^{s-1}dt&=[2]_q^r\sum_{m=0}^{\infty}\frac{\binom{m+r-1}{m}(-1)^m}{[m+x]_q^s}\\&=[2]_q^r\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(-1)^{m_1+\cdots+m_r}}{[m_1+\cdots+m_r+x]_q^s},\ \text{where } s\in\mathbb{C},\ x\neq 0,-1,-2,\cdots.\end{aligned} \tag{17}$$

From (17), we can consider the following multiple q-Euler zeta function.

Definition 2. *For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $x \neq 0, -1, -2, \cdots$, we define the multiple q-Euler zeta function as follows:*

$$\zeta^E_{r,q}(s,x) = [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}}{[m_1+\cdots+m_r+x]^s_q}.$$

Note that $\zeta^E_{r,q}$ is meromorphic function in whole complex s-plane. By using Cauchy residue theorem and Laurent series in (15) and (17), we obtain the following theorem.

Theorem 3. *For $n \in \mathbb{Z}_+$, $r \in \mathbb{N}$, we have*

$$\zeta^E_{r,q}(-n,x) = E^{(r)}_{n,q}(x).$$

In (15), we have

(18)

$$F^{(r)}_q(t,x) = [2]^r_q \sum_{a_1,\cdots,a_r=0}^{f-1} (-1)^{a_1+\cdots+a_r} \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} e^{[\sum_{i=1}^r (a_i+fm_i)+x]_q t}.$$

By (17) and (18), we obtain the following theorem.

Theorem 4. *(Distribution relation for $E^{(r)}_{m,q}(x)$)*
For $n \in \mathbb{Z}_+$, $f \in \mathbb{N}$ with $f \equiv 1$ (mod 2), we have

$$E^{(r)}_{n,q}(x) = \left(\frac{[2]_q}{[2]_{q^f}}\right)^r [f]^n_q \sum_{a_1,\cdots,a_r=0}^{f-1} (-1)^{a_1+\cdots+a_r} E^{(r)}_{n,q^f}\left(\frac{a_1+\cdots+a_r+x}{f}\right).$$

Moreover,

$$E^{(r)}_{n,q}(x) = \frac{[2]^r_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-q^x)^l \sum_{a_1,\cdots,a_r=0}^{f-1} \frac{(-1)^{a_1+\cdots+a_r} q^{l(a_1+\cdots+a_r)}}{(1+q^{lf})^r}.$$

Let χ be the Dirichlet's character with conductor $f(=odd) \in \mathbb{N}$. Then we define the generalized q-Euler polynomials of order r attached to χ as follows:

(19)

$$\begin{aligned} F^{(r)}_{q,\chi}(t,x) &= \sum_{n=0}^{\infty} E^{(r)}_{n,\chi,q}(x) \frac{t^n}{n!} \\ &= [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \left(\prod_{j=1}^{r} \chi(m_j)\right) e^{[x+m_1+\cdots+m_r]_q t}. \end{aligned}$$

In the special case $x = 0$, $E^{(r)}_{n,\chi,q}(= E^{(r)}_{n,\chi,q}(0))$ are called the n-th generalized q-Euler numbers of order r attached to χ. From (19), we can derive
(20)

$$F^{(r)}_{q,\chi}(t,x) = [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \left(\prod_{j=1}^{r} \chi(m_j)\right) e^{[x+m_1+\cdots+m_r]_q t}$$

$$= [2]^r_q \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \sum_{a_1,\cdots,a_r=0}^{f-1} \left(\prod_{j=1}^{r} \chi(a_j)\right) (-1)^{\sum_{j=1}^r a_j} e^{[x+mf+\sum_{j=1}^r a_j]_q t}.$$

By (16) and (20), we obtain the following theorem.

Theorem 5. *For $f \in \mathbb{N}$ with $f \equiv 1(mod\ 2)$, we have*

$$E^{(r)}_{n,\chi,q}(x) = [f]^n_q \left(\frac{[2]_q}{[2]_{q^f}}\right)^r \sum_{a_1,\cdots,a_r=0}^{f-1} \left(\prod_{j=1}^{r} \chi(a_j)\right) (-1)^{\sum_{j=1}^r a_j} E^{(r)}_{n,q^f}\left(\frac{x+\sum_{j=1}^r a_j}{f}\right),$$

and

$$E^{(r)}_{n,\chi,q}(x) = \frac{[2]^r_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-q^x)^l \sum_{a_1,\cdots,a_r=0}^{f-1} \frac{\left(\prod_{j=1}^r \chi(a_j)\right)(-q^l)^{\sum_{i=1}^r a_i}}{(1+q^{lf})^r}.$$

From the Mellin transformation of $F^{(r)}_{q,\chi}(t,x)$, we note that

$$(21) \quad \frac{1}{\Gamma(s)} \int_0^{\infty} F^{(r)}_{q,\chi}(-t,x) t^{s-1} dt = [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}\left(\prod_{j=1}^r \chi(m_j))\right)}{[m_1+\cdots+m_r+x]^s_q},$$

where $s \in \mathbb{C}$, $\Re(x) > 0$. From (21) we can also consider the following Dirichlet's type multiple q-Euler l-function.

Definition 6. *For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $x \neq 0, -1, -2, \cdots$, we define Dirichlet's type q-Euler l-function as follows:*

$$l^{(r)}_q(s,x|\chi) = [2]^r_q \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}\left(\prod_{j=1}^r \chi(m_j)\right)}{[m_1+\cdots+m_r+x]^s_q}.$$

Note that $l^{(r)}_q(s,x|\chi)$ is also holomorphic function in whole complex s-plane. By (20) and (21), we see that

$$l^{(r)}_q(s,x|\chi)$$

$$= \frac{1}{[f]^s_q} \left(\frac{[2]_q}{[2]_{q^f}}\right)^r \sum_{a_1,\cdots,a_r=0}^{f-1} \left(\prod_{j=1}^{r} \chi(a_j)\right) (-1)^{\sum_{i=1}^r a_i} \zeta^E_{r,q^f}\left(s, \frac{a_1+\cdots+a_r+x}{f}\right).$$

By using Laurent series and Cauchy residue theorem, we obtain the following theorem.

Theorem 7. *For $n \in \mathbb{Z}_+$, we have*

$$l_q^{(r)}(-n, x|\chi) = E_{n,\chi,q}^{(r)}(x).$$

For $q = 1$, Theorem 7 seems to be similar type of Dirichlet's L-function in complex analysis. That is, let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$. Then Dirichlet L-function is defined as

$$L(s, x|\chi) = \sum_{n=0}^{\infty} \frac{\chi(n)}{(n+x)^s}, \text{ where } s \in \mathbb{C}, \, x \neq 0, -1, -2, \cdots .$$

Let n be positive integer. Then we have $L(-n, x|\chi) = -\frac{B_{n,\chi}(x)}{n}$, where $B_{n,\chi}(x)$ are called the n-th generalized Bernoulli polynomials attached to χ (see [13, 14, 16, 18, 2, 3, 20-23]).

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